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Entropy of Einstein–Maxwell-Dilaton–Axion Black Holes

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The entropy of Einstein–Maxwell-dilaton–axion black holes is calculated by using the improved brick-wall model. Taking into account of the statistical physics, we propose not to consider the superradiant modes. The result shows that the nonsuperradiant modes do contribute exactly the area-law entropy for extreme black hole. Moreover, our cut-off ε which does not require an angular cut-off δ is independent of angle θ. As for the extreme black hole, we found that its entropy is zero.

KEY WORDS: black hole entropy; brick-wall model; superradiant modes.

In order to give a statistical explanation of black hole entropy, 't Hooft proposed brick-wall method in which the black hole entropy is identified with the entropy of the quantum fields surrounding the black hole itself ('t Hooft, 1985). Since the density of states approaching the horizon diverges, in order to avoid the divergence in the entropy, he has to introduce the cut-off of the order of Plank length, which is interpreted as the position of a "brick wall." Mathematically, the region of nonzero wave function is limited in $r_+ + \varepsilon$ and L, where r_+ is the radius of event horizon and ε, L are ultraviolet cut-off and infrared cut-off, respectively, and $\varepsilon \ll r_+$, $L \gg r_+$. 't Hooft himself studied the contribution to the entropy of Schwarzschild black hole due to scalar field. He found that the leading term of scalar field entropy is one fourth of the area of event horizon. After this, the method was applied to scalar field and neutrino field in various black holes background (Cognola, 1998; Demers *et al.*, 1995; Gao and Shen, 2002; Ghosh and Mitra, 1994; Lee and Kim, 1996; Shen, 2000, 2002; Shen *et al.*, 1997; Shen and Chen, 1999), where it showed that the leading term of neutrino field entropy is seven eighth of the area of event horizon. Recently, the method was extended to the scalar and

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neutrino field in rotating Kerr–Newman black hole space-time (Ho *et al.*, 1997; Liu and Zhao, 2000), where it has been shown that both scalar and neutrino field have two kinds of mode: superradiant and nonsuperradiant modes. The entropy is composed of the superradiant and nonsuperradiant modes. Both modes contribute simultaneously to the entropy with the same order in terms of the cut-off ε. In particular, the contribution of the superradiant mode is negative. To avoid divergence in this method when the angular velocity tends to zero, the authors propose to introduce a lower bound of angular velocity. Moreover, from the lower bound of angular velocity, they obtain the θ dependence structure of cut-off, which naturally requires an angular cut-off δ. Finally, if the cut-off, ε and δ satisfy a proper relation, the entropy satisfies the area law.

In this paper, we give the calculation of entropy for Einstein–Maxwell-dilaton–axion black holes due to scalar and neutrino fields. Here we propose not to consider the entropy of superradiant modes. We consider that the bosons of superradiant modes do not satisfy Bose distribution, while fermions do not display supperradiance (Wald, 1984). Therefore we propose not to consider this modes. In fact, the nonsuperradiant part do exactly give the Bekenstein entropy. Moreover, our cut-off which does not require an angular cut-off δ is independent of angle θ.

We have ever met a difficulty when the brick-wall method was applied to Schwarzschild–de Sitter space-time (Gao and Liu, 2000). Different from non-de Sitter space-time, there are two event horizons in Schwarzschild–de Sitter space-time. The two event horizons have different temperatures. Therefore the radiation between them is not in thermal equilibrium. It is apparent that we should not use the brick-wall model which is based on the thermal equilibrium in a large scale. In other words, the region $r_+ + \varepsilon$ and L is not in thermal equilibrium. Former work tells us that the leading term of entropy in the brick-wall method comes from the contribution of the field very close to the horizon. So, we may assume that there is a thin membrane of quantum fields in the vicinity of event horizon. The distance from the membrane to event horizon is ε and the thickness of the membrane is δ; ε and δ should have the same order, therefore we may assume that the thickness of the membrane is also ε. Thus, the fields in the membrane $[r_+ + \varepsilon, r_+ + 2\varepsilon]$ can be regarded as in locally thermal equilibrium. In fact, Hawking radiation also comes from the vacuum fluctuation in the vicinity of event horizon. Therefore, we might regard the two event horizons as two independent thermal equilibrium sustems and consider them respectively. According to this idea, we improved brick-wall method. Because of technical difficulties, we use this improved method. The computation shows that some mathematical difficulties are greatly decreased by this method

1. SCALAR FIELD

The line element of the Einstein–Maxwell-dilaton–axion space-time is given by (Garcia *et al.*, 1995)

$$
\begin{aligned}
ds^2 &= g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2 \\
&= -\frac{\Delta - a^2\sin^2\theta}{\sum} dt^2 - \frac{2a\sin^2\theta[(r^2+a^2-2Dr)-\Delta]}{\sum} dt\, d\varphi \\
&\quad + \frac{(r^2+a^2-2Dr)^2 - \Delta a^2\sin^2\theta}{\sum}\sin^2\theta\, d\varphi^2 + \frac{\sum}{\Delta} dr^2 + \sum d\theta^2, \quad (1)
\end{aligned}
$$

where $\Delta = r^2 + a^2 - 2Mr$; $\sum = r^2 - 2Dr + a^2\cos^2\theta$; M, a, D are the mass, the specific angular momentum, and the dilaton charge of the black hole.

The wave equation for scalar particles is

$$
\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right) = 0. \quad (2)
$$

The spectrum of Hawking radiation for scalar field in such space-time can be written as

$$
N_\omega^2 = \frac{1}{e^{\beta(\omega-\omega_0)} - 1}, \quad (3)
$$

where N_ω^2, β, ω, ω_0 are the radiation intensity, the inverse of Hawking temperature, the energy and the chemical potential of particles. The ω_0 is defined by

$$
\omega_0 = m\Omega_+, \quad (4)
$$

where m is the magnetic quantum number, $\Omega_+ = \lim_{r\to r_+} \frac{-g_{t\varphi}}{g_{\varphi\varphi}}$ is the angular velocity of event horizon, r_+ is the radius of event horizon.

The distribution of particles corresponding to Eq. (3) is given by

$$
a_l = \frac{\omega_l}{e^{\beta(\omega-\omega_0)} - 1}, \quad (5)
$$

where a_l is the number of particles in the l energy level, ω_l is the degeneracy of l energy level.

Quantum field theory in curved space-time illustrates that there are two modes (superradiant and nonsuperradiant modes) of radiation for bosons in the background of Einstein–Maxwell-dilaton–axion space-time. The superradiant modes are the common feature of rotating black holes and are characterized by modes of $0 \le \omega \le m\Omega_+$ and $m > 0$ (Chandrasekhar, 1983), while the nonsuperradiant modes are those of $\omega > m\Omega_+$ and any m, where ω is the energy of a field particle, m is the azimuthal quantum number, and Ω_+ is the angular velocity of the black hole event horizon. These two kinds of modes are distinct. It is obvious that Eq. (5) is meaningless when $\omega < \omega_0$ for the reason that a_l should not be negative for bosons. In other words, superradiant modes do not satisfy Bose distribution. Therefore, we think that superradiation does not contribute to black hole entropy. As for fermion fields, it does not display superradiance but this does not mean that modes with frequency in the range $0 < \omega < m\Omega_+$ do not exist (Unruh, 1974). We still propose not to consider the "superradiant" part for fermions.

Considering Eq. (5), we set $E = \omega - \omega_0$. Thus the wave function Φ can be written as

$$\Phi(t, r, \theta, \varphi) = e^{-i(E+m\Omega_+)t+im\varphi+iK(r,\theta)}. \tag{6}$$

Substituting Eq. (6) into Eq. (2), we yield the radial wave number

$$k_r^2 = \frac{1}{\Delta^2}\left[\frac{\sum}{\sin^2\theta} g_{ab}T^{ab} - \Delta k_\theta^2\right], \tag{7}$$

where $k_r \equiv \frac{dK}{dr}$ and $k_\theta \equiv \frac{dK}{d\theta}$ are the radial wave number and the angular wave number, respectively. Improved brick-wall method tells us that the black hole entropy mainly comes from the vicinity of event horizon. Considering relations below

$$\lim_{r\to r_+} \frac{g_{t\varphi}}{g_{\varphi\varphi}} = -\Omega_+, \lim_{r\to r_+} \frac{g_{tt}}{g_{\varphi\varphi}} = \Omega_+^2, \tag{8}$$

we have

$$g_{ab}T^{ab} \equiv g_{tt}m^2 + 2g_{t\varphi}(E + m\Omega_+)m + g_{\varphi\varphi}(E + m\Omega_+)^2 \simeq g_{\varphi\varphi}E^2. \tag{9}$$

According to semiclassical quantum theory and improved brick-wall model, the constrain imposed on wave number k reads

$$n_r\pi = \int_{r_++\varepsilon}^{r_++2\varepsilon} dr k_r, \tag{10}$$

where r_+ is the event horizon, ε is a small positive quantity, i.e., $\varepsilon \ll r_+$.

Then free energy can be obtained

$$\beta F = \sum_{mk_\theta n_r} \ln(1 - e^{-\beta E}). \tag{11}$$

It is obvious that Eq. (11) is meaningless when $E < 0$ (for superradiation). This indicates once more the superradiant modes $E < 0$ should be neglected.

The distribution of state density is regarded as being continuous and the free energy is obtained

$$\beta F = \int dm \int dk_\theta \int db_r \ln(1 - e^{-\beta E}) = -\int dm \int dk_\theta \beta \int dE \frac{n_r}{e^{\beta E} - 1}. \tag{12}$$

Substituting Eq. (10) into Eq. (12), we have

$$\begin{aligned}\beta F = -\frac{\beta}{\pi} \int_0^\infty dE \int_{r_++\varepsilon}^{r_++2\varepsilon} dr \int_0^{k_\theta \max} dk_\theta \int_{-k_\theta}^{+k_\theta} dm \\ \times \frac{1}{\Delta}\left[\frac{\sum}{\sin^2\theta} g_{\varphi\varphi}E^2 - \Delta k_\theta^2\right]^{\frac{1}{2}} \frac{1}{e^{\beta E} - 1},\end{aligned} \tag{13}$$

where k_θ is the angular quantum number corresponding to l in spherically symmetric space-time case. In spherical space-time case, the range of m is $-l \leq m \leq l$. Therefore, we choose the range of m being $-k_\theta \leq m \leq k_\theta$ in such case. The extreme of integration in the variable k_θ is due to the fact that $k_{1,2}$ has to be positive. Integrating with respect to m, k_θ, r, respectively, we obtain

$$\beta F = -\frac{\beta}{\pi}\int_0^\infty dE\,\frac{2}{3}\frac{(r_+^2 + a^2 - 2Dr_+)^3}{(r_+ - r_-)^2}\frac{\varepsilon}{\eta^2}\frac{E^3}{e^{\beta E} - 1}. \tag{14}$$

It should be noted that we used the median theorem in the integration with respect to r, hence $\varepsilon < \eta < 2\varepsilon$. We can show the event horizon locates at $\Delta = r^2 + 2Mr - a^2 = 0$, i.e., $(r - r_+)(r - r_-) = 0$, where $r_+ r_-$ are event horizon and the inner horizon, respectively. So, we have $\Delta|_{r_+\eta} = \eta(r_+ - r_-)$. Our cut-off ε is not dependent on θ. Thus, we get the free energy

$$F = -\frac{2\pi^3}{45}\frac{1}{\beta^4}\frac{(r_+^2 + a^2 - 2Dr_+)^3}{(r_+ - r_-)^2}\frac{\varepsilon}{\eta^2}. \tag{15}$$

From the relation between entropy and free energy

$$S = \beta^2\frac{\partial F}{\partial \beta}, \tag{16}$$

we obtain the entropy of the black hole

$$S = \beta^2\frac{\partial F}{\partial \beta} = \frac{8\pi^3}{45}\frac{1}{\beta^3}\frac{(r_+^2 + a^2 - 2Dr_+)^3}{(r_+ - r_-)^2}\frac{\varepsilon}{\eta^2}. \tag{17}$$

ε and η are of the same order, i.e., $\varepsilon \sim \eta$. Thus, we have $\frac{\varepsilon}{\eta^2} \sim \frac{1}{\epsilon}$. Following 't Hooft, we choose $\frac{\varepsilon}{\eta^2} \sim \frac{1}{\epsilon} = 90\beta$. Considering the inverse temperature $\beta = \frac{4\pi(r_+^2 - 2Dr_+ + a^2)}{r_+ - r_-}$ and the area of event horizon $A = 4\pi(r_+^2 + a^2 - 2Dr_+)$, we can rewrite Eq. (17) as

$$S = \frac{1}{4}A. \tag{18}$$

It is exactly the Bekenstein–Hawking entropy.

The result changes if the black hole is extreme:

$$Sextr = \lim_{r_+ \to r_-} S = \frac{8\pi^3}{45}\left[\frac{r_+ r_-}{4\pi(r_+^2 - 2Dr_+ + a^2)^2}\right]^3 \frac{(r_+^2 + a^2 - 2Dr_+)^3}{(r_+ - r_-)^2}\frac{\varepsilon}{\eta^2} = 0$$

Thus, we conclude that the entropy of extreme Einstein–Maxwell-dilaton–axion black hole is zero (Gibbons and Kallosh, 1995). We had better note, we cannot take limit in Eq. (13) for the reason that $\lim_{r_H \to r_-} \beta = \infty$. Therefore, we must take limit in Eq. (17).

2. DIRAC FIELD

In curved space-time, Dirac equation can be written as (Chandrasekhar, 1983)

$$
\begin{aligned}
(D+\varepsilon-\rho)F_1+(\bar{\delta}+\pi-\alpha)F_2 &= \frac{i}{\sqrt{2}}\mu_0 G_1,\\
(\Delta'+\mu-\gamma)F_2+(\delta+\beta-\tau)F_1 &= \frac{i}{\sqrt{2}}\mu_0 G_2,\\
(D+\varepsilon^*-\rho^*)G_2-(\delta+\pi^*-\alpha^*)G_1 &= \frac{i}{\sqrt{2}}\mu_0 F_2,\\
(\Delta'+\mu^*-\gamma^*)G_1-(\bar{\delta}+\beta^*-\tau^*)G_2 &= \frac{i}{\sqrt{2}}\mu_0 F_1,
\end{aligned} \tag{19}
$$

where F_1, F_2, G_1, G_2 are 4-component spinors; μ_0 is the mass of the particle; $\alpha, \beta, \gamma, \varepsilon, \mu, \pi, \rho, \tau$ etc. are Newman–Penrose symbols; while α^*, β^* are, respectively, the complex conjugates of α, β etc., and they are related to the null tetrad as follows

$$
\begin{aligned}
\alpha &= \frac{1}{2}(l_{\mu;\nu}n^\mu \bar{m}^\nu - m_{\mu;\nu}\bar{m}^\mu \bar{m}^\nu), \qquad \rho = l_{\mu;\nu}m^\mu \bar{m}^\nu,\\
\beta &= \frac{1}{2}(l_{\mu;\nu}n^\mu m^\nu - m_{\mu;\nu}\bar{m}^\mu m^\nu), \qquad \pi = -n_{\mu;\nu}\bar{m}^\mu l^\nu,\\
\gamma &= \frac{1}{2}(l_{\mu;\nu}n^\mu n^\nu - m_{\mu;\nu}\bar{m}^\mu n^\nu), \qquad \mu = -n_{\mu;\nu}\bar{m}^\mu m^\nu,\\
\varepsilon &= \frac{1}{2}(l_{\mu;\nu}n^\mu l^\nu - m_{\mu;\nu}\bar{m}^\mu l^\nu), \qquad \tau = l_{\mu;\nu}m^\mu n^\nu,\\
D &= l^\mu\partial_\mu, \quad \Delta' = n^\mu\partial_\mu, \quad \delta = n^\mu\partial_\mu, \quad \bar{\delta} = \bar{m}^\mu\partial_\mu.
\end{aligned} \tag{20}
$$

Choosing the null tetrad below

$$
\begin{aligned}
l^\mu &= \frac{1}{\Delta}(r^2+a^2-2Dr, \Delta, 0, a), \; m^\mu = \frac{1}{\sqrt{2}\bar{\sigma}}\left(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right),\\
n^\mu &= \frac{1}{2\Sigma}(r^2+a^2-2Dr, -\Delta, 0, a), \; \bar{m}^\mu = \frac{1}{\sqrt{2}\bar{\sigma}*}\left(-ia\sin\theta, 0, 1, \frac{-i}{\sin\theta}\right),
\end{aligned} \tag{21}
$$

where $\bar{\sigma}$ is defined by $\bar{\sigma} = \sqrt{r^2-2Dr} + ia\cos\theta$. The corresponding covariant null tetrad is

$$
l_\mu = \frac{1}{\Delta}(\Delta, -\Sigma, 0, -a\Delta\sin^2\theta), \; n_\mu = \frac{1}{2\Sigma}(\Delta, \Sigma, 0, -a\Delta\sin^2\theta),
$$

$$m_\mu = \frac{1}{\sqrt{2}\sigma}[ia\sin\theta, 0, -\Sigma, -i(r^2+a^2-2Dr)\sin\theta],$$
$$\bar{m}^\mu = -\frac{1}{\sqrt{2}\sigma*}[-ia\sin\theta, 0, -\Sigma, i(r^2+a^2-2Dr)\sin\theta], \tag{22}$$

Considering the symmetry of space-time, we set

$$F_1 = e^{-i(E+m\Omega_+)t+im\varphi}\left(\sqrt{r^2-2Dr}+ia\cos\theta\right)^{-1} f_1(r,\theta),$$
$$F_2 = e^{-i(E+m\Omega_+)t+im\varphi} f_2(r,\theta),\ G_1 = e^{-i(E+m\Omega_+)t+im\varphi} g_1(r,\theta),$$
$$G_2 = e^{-i(E+m\Omega_+)t+im\varphi}(\sqrt{r^2-2Dr}+ia\cos\theta)^{-1} g_2(r,\theta), \tag{23}$$

Then Eq. (19) can be written as

$$D_0 f_1 + \frac{1}{\sqrt{2}}L_- f_2 = \frac{1}{\sqrt{2}}\left(i\mu_0\sqrt{r^2-2Dr}+a\mu_0\cos\theta\right) g_1,$$
$$\Delta D_1 f_2 - \sqrt{2}L_+ f_1 = -\sqrt{2}\left(i\mu_0\sqrt{r^2-2Dr}+a\mu_0\cos\theta\right) g_2,$$
$$D_0 f_2 - \frac{1}{\sqrt{2}}L_+ g_1 = \frac{1}{\sqrt{2}}\left(i\mu_0\sqrt{r^2-2Dr}-a\mu_0\cos\theta\right) f_2,$$
$$\Delta D_1 g_1 + \sqrt{2}L_- g_2 = -\sqrt{2}\left(i\mu_0\sqrt{r^2-2Dr}-a\mu_0\cos\theta\right) f_1, \tag{24}$$

where

$$D_0 = -\frac{\partial}{\partial r} - \frac{iP}{\Delta},\ D_1 = \frac{\partial}{\partial r} + \frac{iP}{\Delta} + 2\frac{r-M}{r},\ L_\pm = \frac{\partial}{\partial\theta} \pm q + \frac{1}{2}ctg\theta,$$
$$q = a(E+m\Omega_+)\sin\theta - \frac{m}{\sin\theta},\ P = (r^2+a^2-2Dr)(E+m\Omega_+) - am. \tag{25}$$

Making further transformations

$$f_1 = R_-(r)S_-(\theta), \qquad g_1 = R_+(r)S_-(\theta),$$
$$f_2 = R_+(r)S_+(\theta), \qquad g_2 = R_-(r)S_+(\theta),$$

we obtain the radial equations and angular equations (for simplicity we set $\mu_0 = 0$)

$$\Delta D_1 D_0 R_- = \lambda^2 R_-, \qquad L_- L_+ S_- = -\lambda^2 S_-,$$
$$D_0(\Delta D_1 R_+) = \lambda^2 R_+, \qquad L_+ L_- S_+ = -\lambda^2 S_+, \tag{26}$$

Using WKB approximation i.e., setting $R_{\pm} = e^{K_{\pm}}$, we obtain the wave numbers

$$k_{+} = \frac{r^2 + a^2 - 2Dr}{\Delta}\sqrt{E^2 - \frac{\Delta(\lambda^2 - 2)}{(r^2 + a^2 - 2Dr)^2}}, \tag{27}$$

$$k_{-} = \frac{r^2 + a^2 - 2Dr}{\Delta}\sqrt{E^2 - \frac{\Delta\lambda^2}{(r^2 + a^2 - 2Dr)^2}}, \tag{28}$$

where λ^2 is the separation constant. We had better note that in the vicinity of event horizon Eq. (7) becomes

$$k_{r} = \frac{r^2 + a^2 - 2Dr}{\Delta}\sqrt{E^2 - \frac{\Delta k_\theta^2}{(r^2 + a^2 - 2Dr)^2}}. \tag{29}$$

Comparing Eqs. (28), (29) with Eq. (29), we find that they are of the same form. Therefore, we conclude that according to the improved brick-wall model two method of separation variables [Eqs. (6) and (23)] are identical.

Completely similar to the scalar field, we obtain the black hole entropy S due to Dirac particles

$$S = \frac{7}{2}\frac{7}{4}A. \tag{30}$$

In summary, we calculated the entropy of Einstein–Maxwell-dilaton–axion black holes due to bosons and fermions. We consider that the bosons of superradiant modes do not satisfy Bose distribution; while fermions do not display superradiance. Therefore, we propose not to consider this mode. The result shows that the nonsuperradiant modes do contribute exactly the Bekenstein–Hawking entropy. Moreover, our cut-off ε which does not require an angular cut-off δ is independent of angle θ. As for the extreme black hole, we found that its entropy is zero according to brick-wall model.

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